

WELL-POSEDNESS FOR THE SUPERCRITICAL GKDV EQUATION

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ABSTRACT. In this paper we consider the supercritical generalized Korteweg-de Vries equation $\partial_t \psi + \partial_{xxx} \psi + \partial_x (|\psi|^{p-1} \psi) = 0$, where $5 \leq p \in \mathbb{R}$. We prove a local well-posedness result in the homogeneous Besov space $\dot{B}_{\infty}^{s_p, 2}(\mathbb{R})$, where $s_p = \frac{1}{2} - \frac{2}{p-1}$ is the scaling critical index. In particular local well-posedness in the smaller inhomogeneous Sobolev space $H^{s_p}(\mathbb{R})$ can be proved similarly. As a byproduct a global well-posedness result for small initial data is also obtained.

1. INTRODUCTION

Consider the initial value problem associated to the generalized Korteweg-de Vries (gKdV) equation, that is

$$\begin{cases} \partial_t \psi + \partial_{xxx} \psi + \partial_x (|\psi|^{p-1} \psi) &= 0, \\ \psi(0, x) &= \psi_0(x). \end{cases} \quad (1)$$

Well-posedness results of the Cauchy problem (1) (with $p \geq 2$) has been studied by many authors in recent years. We want to give a brief overview of the best known well-posedness results.

The fundamental work on this topic was done by Kenig, Ponce and Vega [6, 7] in 1993 and 1996. They proved local and small data global well-posedness for the sub-critical cases $p \in \{2, 3, 4\}$ in $H^s(\mathbb{R})$ for certain s . For the KdV equation ($p = 2$) they proved well-posedness for $s > -\frac{3}{4}$. In the limiting case $s = -\frac{3}{4}$ existence of solutions has been obtained by Christ, Colliander, and Tao [2]. Kenig, Ponce and Vega also proved well-posedness of the mKdV equation ($p = 3$) for $s \geq \frac{1}{4}$, and of the quartic gKdV equation ($p = 4$) for $s \geq \frac{1}{12}$. So far the scaling space H^{s_p} with $s_p = \frac{1}{2} - \frac{2}{p-1}$ was not reached for the sub-critical cases. That changed in 2007, when Tao [12] proved local well-posedness (and global well-posedness for small data) of the quartic KdV equation in the scaling critical inhomogeneous Sobolev space $\dot{H}^{-\frac{1}{6}}$. In 2012 Koch and Marzuola [8] simplified and strengthened Tao's well-posedness result in the Besov space $\dot{B}_{\infty}^{-\frac{1}{6}, 2}$. For the supercritical cases $p \geq 5$, $p \in \mathbb{N}$, local well-posedness and global well-posedness for small data in the

scaling critical spaces \dot{H}^{s_p} was obtained by Kenig, Ponce and Vega in 1993. Recently Farah, Linares and Pastor extended the global well-posedness result for $p \geq 5$, $p \in \mathbb{N}$. In 2003 Molinet and Ribaud [11] extended the well-posedness result in the supercritical cases to the homogeneous Besov space $\dot{B}_{\infty}^{s_p,2}(\mathbb{R})$ with integer p . To our knowledge well-posedness results for non-integer $p \geq 5$ were not obtained so far. We present a unified proof of well-posedness in the homogeneous Besov space $\dot{B}_{\infty}^{s_p,2}(\mathbb{R})$ for all $5 \leq p \in \mathbb{R}$.

In this paper we pick up techniques of Koch and Marzuola [8] to prove local (and small data global) well-posedness for the supercritical gKdV equation, i.e. (1) with $5 \leq p \in \mathbb{R}$. The well-posedness is proved in the homogeneous Besov space $\dot{B}_{\infty}^{s_p,2}(\mathbb{R})$ (see Definition 2.10), where

$$s_p = \frac{1}{2} - \frac{2}{p-1}$$

is the scaling critical exponent. The homogeneous Besov space $\dot{B}_{\infty}^{s_p,2}(\mathbb{R})$ is slightly larger than the scaling invariant homogeneous Sobolev space $\dot{H}^{s_p}(\mathbb{R})$ consisting of all functions u such that

$$\|u(t)\|_{\dot{H}^{s_p}} = \left(\sum_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^{2s_p} \|u_{\lambda}(t)\|_{L^2}^2 \right)^{1/2} < \infty.$$

Here, u_{λ} denotes the Littlewood-Paley decomposition of u at frequency λ that is defined in Section 2.

In the following, let v be a solution to the Airy equation with same initial data

$$\begin{cases} \partial_t v + \partial_{xxx} v &= 0, \\ v(0, x) &= \psi_0(x). \end{cases} \quad (2)$$

For the quartic gKdV equation

$$\begin{cases} \partial_t \psi + \partial_{xxx} \psi + \partial_x(\psi^4) &= 0, \\ \psi(0, x) &= \psi_0(x), \end{cases} \quad (3)$$

Koch and Marzuola [8] proved the following local well-posedness result:

Theorem 1.1 (Koch and Marzuola [8]). *Let $r_0 > 0$. Then there exist $\varepsilon_0, \delta_0 > 0$ such that, if $0 < T \leq \infty$,*

$$\|\psi_0\|_{\dot{B}_{\infty}^{-\frac{1}{6},2}} \leq r_0$$

and

$$\sup_{\lambda \in 1.01^{\mathbb{Z}}} \|v_{\lambda}\|_{L^6([0,T],\mathbb{R})} \leq \delta_0$$

then there is an unique solution $\psi = v + w$ to (3) with

$$\|w\|_{\dot{X}_{\infty,T}^{-\frac{1}{6}}} \leq \varepsilon_0.$$

Moreover, the function w (and hence ψ) depends analytically on the initial data.

From this local well-posedness result they even obtained global well-posedness for small data ψ_0 , since one easily proves by Strichartz' estimates and the definition of the spaces that

$$\sup_{\lambda \in 1.01\mathbb{Z}} \|v_\lambda\|_{L^6([0,T],\mathbb{R})} \lesssim \|\psi_0\|_{\dot{B}_{\infty}^{-\frac{1}{6},2}}.$$

In the sequel, we are going to prove the analogue statement in the supercritical case, i.e. for (1) with $5 \leq p \in \mathbb{R}$:

Theorem 1.2. *Let $5 \leq p \in \mathbb{R}$, $s_p = \frac{1}{2} - \frac{2}{p-1}$ and $r_0 > 0$. Then there exist $\varepsilon_0, \delta_0 > 0$ such that, if $0 < T \leq \infty$,*

$$\|\psi_0\|_{\dot{B}_{\infty}^{s_p,2}} \leq r_0$$

and

$$\sup_{\lambda \in 1.01\mathbb{Z}} \lambda^{\frac{1}{6}+s_p} \|v_\lambda\|_{L^6([0,T],\mathbb{R})} \leq \delta_0, \quad (4)$$

then there exists an unique solution $\psi = v + w$ to (1) with

$$\|w\|_{\dot{X}_T^{s_p}} \leq \varepsilon_0.$$

Moreover, the solution map is Lipschitz continuous.

Using the same arguments as Koch and Marzuola, we obtain global well-posedness for small initial data ψ_0 as well:

Corollary 1.3. *Let $5 \leq p \in \mathbb{R}$, $s_p = \frac{1}{2} - \frac{2}{p-1}$ and $\delta_0(1)$ be the δ_0 of Theorem 1.2, which depends on r_0 , evaluated at $r_0 = 1$. Let κ_0 and κ_1 be the constants from Lemma 3.1 and Lemma 2.12, respectively. Then there exists $\varepsilon_0 > 0$ such that for*

$$\|\psi_0\|_{\dot{B}_{\infty}^{s_p,2}} \leq \min \left\{ 1, \frac{\delta_0(1)}{\kappa_0 \kappa_1} \right\}$$

there is an unique solution $\psi = v + w$ to (1) with

$$\|w\|_{\dot{X}^{s_p}} \leq \varepsilon_0.$$

Moreover, the solution map is Lipschitz continuous.

The main ingredient of the proof of Theorem 1.2 is a multi-linear estimate that gives bounds on the Duhamel term of the nonlinearity. A crucial tool to get these estimates are the recently introduced U^p and V^p spaces. The rest of the proof is a standard fixed point argument to get

existence and uniqueness. However, due to the non-integer exponents, this argument gets a bit more delicate.

Remark 1. The analogue local and global well-posedness in the inhomogeneous Sobolev space H^{s_p} follows along these lines. Note that the function spaces and the summation has to be modified.

Remark 2. It is possible to choose different Hölder exponents in the proof of the multi-linear estimates (Lemma 4.1 and Lemma 4.2) and hence to require an other smallness condition replacing the smallness condition (4) of the linear solution.

Throughout this paper, we will use mixed Lebesgue spaces $L_t^p L_x^q$ which are defined via the norm

$$\|f\|_{L_t^p L_x^q} = \left(\int \|f(t, \cdot)\|_{L_x^q}^p dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 1 \leq q \leq \infty,$$

and with obvious modifications for $p = \infty$. If $p = q$, then we write $L_{t,x}^p$ for brevity. Moreover, we want to mention that we write $A \lesssim B$, if there is a harmless constant $c > 0$ such that $A \leq cB$.

This paper is organized as follows: In Section 2 we give a brief introduction to the function spaces used in this paper. Section 3 provides some basic linear and bilinear estimates. Multi-linear estimates to control the Duhamel term of the nonlinearity are proved in Section 4. Theorem 1.2 and the global well-posedness result is proved in Section 5.

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2. FUNCTION SPACES

Crucial tools to prove this well-posedness results are the function spaces U^p , which have been introduced in the context of dispersive PDEs by Tataru and Koch-Tataru [9, 10] as well as the closely related spaces of bounded p -Variation V^p due to Wiener [13]. The following exposition of the U^p and V^p spaces may be found in [5]. We refer the reader to this paper for detailed definitions and proofs.

We consider functions taking values in $L^2 = L^2(\mathbb{R}^d, \mathbb{R})$, but in the general part of this section one may replace L^2 by an arbitrary Hilbert space. Let \mathcal{Z} be the set of finite partitions $-\infty < t_0 < t_1 < \dots < t_K \leq \infty$.

Definition 2.1. Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$ and $\phi_0 = 0$, we call the function $a : \mathbb{R} \rightarrow L^2$ given by

$$a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \phi_{k-1}$$

a U^p -atom. Furthermore, we define the atomic space

$$U^p = \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ } U^p\text{-atom, } \lambda_j \in \mathbb{C}, \text{ s.t. } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

endowed with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j \in \mathbb{C}, \text{ s.t. } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}.$$

Two useful statements about U^p are collected in the following

Proposition 2.2. *Let $1 \leq p < q < \infty$.*

- (i) $\|\cdot\|_{U^p}$ is a norm. The space U^p is complete and hence a Banach space.
- (ii) The embeddings $U^p \subset U^q \subset L^\infty(\mathbb{R}, L^2)$ are continuous.

Definition 2.3. Let $1 \leq p < \infty$.

- (i) We define V^p as the normed space of all functions $v : \mathbb{R} \rightarrow L^2$ such that $\lim_{t \rightarrow \pm\infty} v(t)$ exists and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}}$$

is finite. We use the convention that $v(-\infty) = \lim_{t \rightarrow -\infty} v(t)$ and $v(\infty) = 0$.

- (ii) We denote the closed subspace of all right-continuous functions $v : \mathbb{R} \rightarrow L^2$ such that $\lim_{t \rightarrow -\infty} v(t) = 0$ by V_{rc}^p .

Remark 3. Note that we set $v(\infty) = 0$, which may differ from the limit of v at ∞ .

Proposition 2.4. *Let $1 \leq p < q < \infty$.*

- (i) The embedding $U^p \subset V_{rc}^p$ is continuous.
- (ii) The embeddings $V^p \subset V^q$ are continuous.
- (iii) The embedding $V_{rc}^p \subset U^q$ is continuous, and

$$\|v\|_{U^q} \leq c_{p,q} \|v\|_{V^p}.$$

Proposition 2.5. *For $u \in U^p$ and $v \in V^{p'}$, where $1 = \frac{1}{p} + \frac{1}{p'}$, and a partition $\mathbf{t} := \{t_k\}_{k=0}^K \in \mathcal{Z}$ we define*

$$B_{\mathbf{t}}(u, v) := \sum_{k=1}^K \langle u(t_{k-1}), v(t_k) - v(t_{k-1}) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of L^2 . Notice that $v(t_K) = 0$ since $t_K = 0$ for all partitions $\{t_k\}_{k=0}^K \in \mathcal{Z}$. There is a unique number $B(u, v)$ with the property that for all $\varepsilon > 0$ there exists $\mathbf{t} \in \mathcal{Z}$ such that for every $\mathbf{t}' \subset \mathbf{t}$ it holds

$$|B_{\mathbf{t}'}(u, v) - B(u, v)| < \varepsilon,$$

and the associated bilinear form

$$B : U^p \times V^{p'} : (u, v) \mapsto B(u, v)$$

satisfies the estimate

$$|B(u, v)| \leq \|u\|_{U^p} \|v\|_{V^{p'}}.$$

Proposition 2.6. *Let $1 < p < \infty$. We have*

$$(U^p)^* = V^{p'}$$

in the sense that

$$T : V^{p'} \rightarrow (U^p)^*, \quad T(v) := B(\cdot, v)$$

is an isometric isomorphism.

Corollary 2.7. *For $1 < p < \infty$, $u \in U^p$ and for $v \in V^{p'}$ the following estimates hold true*

$$\|u\|_{U^p} = \sup_{\substack{v \in V^{p'} \\ \|v\|_{V^{p'}}=1}} |B(u, v)|$$

and

$$\|v\|_{V^{p'}} = \sup_{u \text{ } U^{p'}\text{-atom}} |B(u, v)|.$$

Proposition 2.8. *Let $1 < p < \infty$. If the distributional derivative of u is in L^1 and $v \in V^p$. Then,*

$$B(u, v) = - \int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle dt.$$

Following Bourgain's strategy for the Fourier restriction spaces we adapt the U^p and V^p space to the gKdV equation.

Definition 2.9. Define the Airy group $S : C(\mathbb{R}, L^2) \rightarrow C(\mathbb{R}, L^2)$ as

$$S(t) := e^{-it\partial_x^3} = \mathcal{F}_x^{-1} e^{-it\xi^3} \mathcal{F}_x,$$

where \mathcal{F}_x denotes the Fourier transform with respect to x . For $u \in C(\mathbb{R}, L^2)$ we set $v(t) := S(-t)u(t)$ and define

$$U_{\text{KdV}}^p := SU^p \quad \text{and} \quad V_{\text{KdV}}^p := SV_{rc}^p,$$

with norms

$$\|u\|_{U_{\text{KdV}}^p} = \|v\|_{U^p} \quad \text{and} \quad \|u\|_{V_{\text{KdV}}^p} = \|v\|_{V^p}.$$

Again, we define a bilinear map B_{KdV} such that for $u \in U_{\text{KdV}}^p$, $v \in V_{\text{KdV}}^{p'}$, we have for a function u with $(\partial_t + \partial_x^3)u \in L^1 L^2$

$$B_{\text{KdV}}(u, v) = - \int \langle (\partial_t + \partial_x^3)u, v \rangle dt.$$

By this bilinear map, we obtain similar duality statements as in Corollary 2.7.

For minor technical purposes we use a slight unusual Littlewood-Paley decomposition, using powers of 1.01 instead of 2 (cf. [8, 12]): We fix a nonnegative, even function $\phi \in C_0^\infty((-2, 2))$ with $\phi(s) = 1$ for $|s| \leq 1$. We use this function to define a partition of unity: for $\lambda \in 1.01^{\mathbb{Z}}$, we set

$$\Psi_\lambda(\xi) = \phi\left(\frac{|\xi|}{\lambda}\right) - \phi\left(\frac{1.01|\xi|}{\lambda}\right).$$

We define the Littlewood-Paley operators $P_\lambda : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ as the Fourier multiplier with symbol Ψ_λ . For brevity we write $u_\lambda := P_\lambda u$. Furthermore, we define

$$u_{\leq \lambda} := P_{\leq \lambda} u := \sum_{\substack{\mu \leq \lambda \\ \mu \in 1.01^{\mathbb{Z}}}} P_\mu u \quad \text{and} \quad u_{< \lambda} := P_{< \lambda} u := (P_{\leq \lambda} - P_\lambda)u.$$

Definition 2.10. For $s \in \mathbb{R}$, we define the homogeneous Besov spaces $\dot{B}_\infty^{s,2}$ as the set of all tempered distributions on \mathbb{R}^n for which the norm

$$\|v\|_{\dot{B}_\infty^{s,2}} = \sup_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^s \|v_\lambda\|_{L_x^2}$$

is finite.

We pick up the homogeneous space \dot{X}^s that was defined in [8].

Definition 2.11. For $s \in \mathbb{R}$, we define the real-valued homogeneous space \dot{X}^s using the norm

$$\|v\|_{\dot{X}^s} = \sup_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^s \|v_\lambda\|_{V_{\text{KdV}}^2}.$$

Furthermore, we denote by \dot{X}_T^s the functions on the time space set $(0, T) \times \mathbb{R}$.

The following estimate follows directly from the definition of the spaces U_{KdV}^p and V_{KdV}^p .

Lemma 2.12. *Let v be a solution to the Airy equation*

$$\begin{cases} \partial_t v + \partial_{xxx} v &= f, \\ v(0, x) &= v_0(x), \end{cases}$$

then, for $s \in \mathbb{R}$, there exists $\kappa_1 > 0$ such that the following estimate holds true

$$\|v\|_{\dot{X}_T^s} \leq \kappa_1 \left(\|v_0\|_{\dot{B}_{\infty}^{s,2}} + \sup_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^s \left\| \int_0^t S(t-s) f_{\lambda}(s) ds \right\|_{U_{\text{KdV}}^2} \right).$$

3. LINEAR AND BILINEAR ESTIMATES

The following Lemma is based on [6] and may be found in [8, formula (3.2) and (7.7)].

Lemma 3.1 (Strichartz' estimates). *Let $u \in U_{\text{KdV}}^q$, $q > 4$ and (q, r) be a Strichartz pair of the Airy equation, i.e. $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. Then,*

$$\|u\|_{L_t^q L_x^r} \lesssim \| |D_x|^{-\frac{1}{q}} u \|_{U_{\text{KdV}}^q}. \quad (5)$$

In particular, for $\lambda \in 1.01^{\mathbb{Z}}$ we have

$$\|u_{\lambda}\|_{L_t^q L_x^r} \lesssim \lambda^{-\frac{1}{q}} \|u_{\lambda}\|_{U_{\text{KdV}}^q} \lesssim \lambda^{-\frac{1}{q}} \|u_{\lambda}\|_{V_{\text{KdV}}^2}.$$

Hence for $s \in \mathbb{R}$ it holds that

$$\sup_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^{\frac{1}{q}+s} \|u_{\lambda}\|_{L_t^q L_x^r} \leq \kappa_0 \|u\|_{\dot{X}_T^s}.$$

Lemma 3.2 ([8, page 179]). *Let $u \in \dot{X}_T^{s_p}$, $\lambda \in 1.01^{\mathbb{Z}}$, then we have for all $p \geq 5$*

$$\|u_{\leq \lambda}\|_{L_{t,x}^{\infty}} \lesssim \lambda^{\frac{1}{2}-s_p} \|u\|_{\dot{X}_T^{s_p}}.$$

Proof. This estimate follows directly from Bernstein's inequality and the energy estimate. \square

The next Corollary immediately follows from interpolating the $L_{t,x}^6$ -Strichartz estimate and the $L_{t,x}^{\infty}$ estimate.

Corollary 3.3. *Let $u \in V_{\text{KdV}}^2$, $\lambda \in 1.01^{\mathbb{Z}}$ and $q \geq 6$, then we have*

$$\|u_{\lambda}\|_{L_{t,x}^q} \lesssim \lambda^{\frac{1}{2}-\frac{4}{q}} \|u_{\lambda}\|_{V_{\text{KdV}}^2},$$

and if $p \geq 5$, then we even have for all $q > 2(p-1)$

$$\|u_{\leq \lambda}\|_{L_{t,x}^q} \lesssim \lambda^{\frac{1}{2}-\frac{4}{q}-s_p} \|u_{\leq \lambda}\|_{\dot{X}_T^{s_p}}.$$

The following bilinear estimate is based on a bilinear estimate of Grünrock [4] and can be found in [8, formula (7.8)].

Lemma 3.4 (Bilinear estimate). *Let $u, v \in U_{\text{KdV}}^2$ and let $\lambda, \mu \in 1.01^{\mathbb{Z}}$ such that $\lambda \geq 1.1\mu$. Then*

$$\|v_\mu u_\lambda\|_{L_{t,x}^2} \lesssim \lambda^{-1} \|v_\mu\|_{U_{\text{KdV}}^2} \|u_\lambda\|_{U_{\text{KdV}}^2}.$$

Corollary 3.5. *Let $2 < q \leq \infty$ and $\lambda \geq 1.1\mu$. Then for $u, v \in \dot{X}_T^{s_p}$*

$$\|v_\mu u_\lambda\|_{L_{t,x}^q} \lesssim \mu^{\frac{1}{2}-\frac{1}{q}-s_p} \lambda^{\frac{1}{2}-\frac{3}{q}-s_p} \|v\|_{\dot{X}_T^{s_p}} \|u\|_{\dot{X}_T^{s_p}}.$$

If in addition $p \geq 5$, then for all $q > \frac{p-1}{2}$ we may even estimate

$$\|v_{\leq \mu} u_\lambda\|_{L_{t,x}^q} \lesssim \mu^{\frac{1}{2}-\frac{1}{q}-s_p} \lambda^{\frac{1}{2}-\frac{3}{q}-s_p} \|v\|_{\dot{X}_T^{s_p}} \|u\|_{\dot{X}_T^{s_p}}. \quad (6)$$

Proof. The first inequality follows by interpolating the bilinear estimate (Lemma 3.4) and the $L_{t,x}^\infty$ estimate (Lemma 3.2) as well as Proposition 2.4. As a consequence the second inequality simply follows from a Littlewood-Paley decomposition of $v_{\leq \mu}$. Note that in (6) q is chosen such that the exponent of μ is larger than zero. \square

4. MULTI-LINEAR ESTIMATES

Lemma 4.1. *Let $5 \leq p \in \mathbb{R}$ and $\lambda_2 \leq \dots \leq \lambda_5 \in 1.01^{\mathbb{Z}}$, $\mu \in 1.01^{\mathbb{Z}}$ and $1.1\lambda_5 > \mu$. There exists $r > 0$ independent of T such that for given $v_i, u \in \dot{X}_T^{s_p}$, $i = 0, \dots, 5$, we have for some small $\varepsilon > \delta > 0$*

$$\begin{aligned} & \left| \int |v_{0,\leq \lambda_2}|^{p-5} v_{1,\leq \lambda_2} v_{2,\lambda_2} \cdots v_{5,\lambda_5} u_\mu dx dt \right| \\ & \leq r \lambda_2^\delta \lambda_5^{-\varepsilon-s_p} \mu^{-1-\delta+\varepsilon} \|v_0\|_{\dot{X}_T^{s_p}}^{p-5} \prod_{i=1}^5 \|v_i\|_{\dot{X}_T^{s_p}} \|u_\mu\|_{V_{\text{KdV}}^2}. \end{aligned}$$

Moreover, we may even replace one factor $\|v_i\|_{\dot{X}_T^{s_p}}$, $i = 3, 4$, on the right hand side by

$$\sup_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^{\frac{1}{6}+s_p} \|v_{i,\lambda}\|_{L^6([0,T],\mathbb{R})}.$$

Proof. In order to prove this multi-linear estimate, we distinguish two cases. First, we consider the case when all frequencies λ_i are comparable, i.e. $\lambda_5 \leq 1.1\lambda_2$. In the second case, we consider the situation if the

frequency λ_5 is much greater than λ_2 , i.e. $1.1\lambda_2 < \lambda_5$. In this situation, we can make use of the strong bilinear estimate.

1st case: $1.1\lambda_2 \geq \lambda_5$

We start by assuming that $p > 5$ and consider the integral

$$\left| \int |v_{0,\leq\lambda_2}|^{p-5} v_{1,\leq\lambda_2} v_{2,\lambda_2} \cdots v_{5,\lambda_5} u_\mu dx dt \right|.$$

Let $q = 2(p - 4)$, then using Hölder's inequality we can estimate the integral by

$$\|v_{0,\leq\lambda_2}\|_{L_t^\infty L_x^q}^{p-5} \|v_{1,\leq\lambda_2}\|_{L_t^\infty L_x^q} \|v_{2,\lambda_2}\|_{L_t^{9/2} L_x^{18}} \|v_{3,\lambda_3}\|_{L_{t,x}^6} \|v_{4,\lambda_4}\|_{L_{t,x}^6} \|v_{5,\lambda_5}\|_{L_t^{9/2} L_x^{18}} \|u_\mu\|_{L_t^{9/2} L_x^{18}}.$$

By Sobolev embeddings, the energy estimate and the definition of $\dot{X}_T^{s_p}$, we obtain

$$\|v_{i,\leq\lambda_2}\|_{L_t^\infty L_x^q} \lesssim \lambda_2^{\frac{1}{2}-\frac{1}{q}-s_p} \|v_i\|_{\dot{X}_T^{s_p}}, \quad i = 0, 1.$$

Strichartz' estimates allows to determine

$$\begin{aligned} \|v_{i,\lambda_i}\|_{L_t^{9/2} L_x^{18}} &\lesssim \lambda_i^{-\frac{2}{9}-s_p} \|v_i\|_{\dot{X}_T^{s_p}}, \quad i = 2, 5, \\ \|v_{i,\lambda_i}\|_{L_{t,x}^6} &\lesssim \lambda_i^{-\frac{1}{6}-s_p} \|v_i\|_{\dot{X}_T^{s_p}}, \quad i = 3, 4, \end{aligned}$$

as well as

$$\|u_\mu\|_{L_t^{9/2} L_x^{18}} \lesssim \mu^{-\frac{2}{9}} \|u_\mu\|_{V_{\text{KdV}}^2}.$$

Since λ_2 and λ_5 are comparable, the product of the terms of λ_i can be estimated by a constant times $\lambda_2^{\frac{2}{9}} \lambda_5^{-1-s_p}$ for instance.

If $p = 5$, then we split the integral into two terms, namely

$$I_1 + I_2 = \left| \int v_{1,\ll\lambda_2} v_{2,\lambda_2} \cdots v_{5,\lambda_5} u_\mu dx dt \right| + \left| \int v_{1,\sim\lambda_2} v_{2,\lambda_2} \cdots v_{5,\lambda_5} u_\mu dx dt \right|,$$

where $v_{1,\ll\lambda_2} := \sum_{\lambda_1: 1.1\lambda_1 < \lambda_2} v_{1,\lambda_1}$ and $v_{1,\sim\lambda_2} := v_{1,\leq\lambda_2} - v_{1,\ll\lambda_2}$. Using Hölder's inequality we may estimate I_1 and obtain

$$\|v_{1,\ll\lambda_2} v_{2,\lambda_2}\|_{L_{t,x}^{5/2}} \|v_{3,\lambda_3}\|_{L_{t,x}^{10}} \|v_{4,\lambda_4}\|_{L_{t,x}^6} \|v_{5,\lambda_5}\|_{L_{t,x}^6} \|u_\mu\|_{L_{t,x}^6}.$$

Applying the bilinear estimate, Corollary 3.3 and Strichartz' estimates yields

$$\begin{aligned} \|v_{1,\leq\lambda_2} v_{2,\lambda_2}\|_{L_{t,x}^{5/2}} &\lesssim \lambda_2^{-\frac{3}{5}} \|v_1\|_{\dot{X}_T^0} \|v_2\|_{\dot{X}_T^0}, \\ \|v_{3,\lambda_3}\|_{L_{t,x}^{10}} &\lesssim \lambda_3^{\frac{1}{10}} \|v_3\|_{\dot{X}_T^0}, \\ \|v_{i,\lambda_i}\|_{L_{t,x}^6} &\lesssim \lambda_i^{-\frac{1}{6}} \|v_i\|_{\dot{X}_T^0}, \quad i = 4, 5, \\ \|u_\mu\|_{L_{t,x}^6} &\lesssim \mu^{-\frac{1}{6}} \|u_\mu\|_{V_{\text{KdV}}^2}. \end{aligned}$$

I_2 simply can be estimated by

$$\|v_{1,\sim\lambda_2}\|_{L_{t,x}^6} \|v_{2,\lambda_2}\|_{L_{t,x}^6} \|v_{3,\lambda_3}\|_{L_{t,x}^6} \|v_{4,\lambda_4}\|_{L_{t,x}^6} \|v_{5,\lambda_5}\|_{L_{t,x}^6} \|u_\mu\|_{L_{t,x}^6}.$$

using Hölder's inequality, which can be further estimated by Strichartz' estimates. Since $1.1\lambda_2 \geq \lambda_5$, the product of the λ_i frequencies can be estimated by, e.g., $\lambda_2^{\frac{1}{6}} \lambda_5^{-1}$.

2nd case: $1.1\lambda_2 < \lambda_5$

The main idea in this situation is to use the strong bilinear estimate, which allows to bound

$$\|v_{2,\lambda_2} v_{5,\lambda_5}\|_{L_{t,x}^q} \lesssim \lambda_2^{\frac{1}{2}-\frac{1}{q}} \lambda_5^{\frac{1}{2}-\frac{3}{q}} \|v_2\|_{\dot{X}_T^{s_p}} \|v_5\|_{\dot{X}_T^{s_p}}$$

provided $2 < q \leq \infty$. We define the Hölder exponents $q_1 = 2(p+5)$ and $q_2 = 2 + \frac{2}{p+4}$. Using Hölder's inequality, we may bound

$$\left| \int |v_{0,\leq\lambda_2}|^{p-5} v_{1,\leq\lambda_2} v_{2,\lambda_2} \cdots v_{5,\lambda_5} u_\mu dx dt \right|$$

by

$$\|v_{0,\leq\lambda_2}\|_{L_{t,x}^\infty}^{p-5} \|v_{1,\leq\lambda_2}\|_{L_{t,x}^{q_1}} \|v_{3,\lambda_3}\|_{L_{t,x}^6} \|v_{4,\lambda_4}\|_{L_{t,x}^6} \|u_\mu\|_{L_{t,x}^6} \|v_{2,\lambda_2} v_{5,\lambda_5}\|_{L_{t,x}^{q_2}}.$$

From Lemma 3.2 and the definition of $\dot{X}_T^{s_p}$, we obtain

$$\|v_{0,\leq\lambda_2}\|_{L_{t,x}^\infty}^{p-5} \lesssim \lambda_2^{(p-5)(\frac{1}{2}-s_p)} \|v_0\|_{\dot{X}_T^{s_p}}^{p-5}.$$

Corollary 3.3 allows to estimate

$$\|v_{1,\leq\lambda_2}\|_{L_{t,x}^{q_1}} \lesssim \lambda_2^{\frac{1}{2}-\frac{4}{q_1}-s_p} \|v_1\|_{\dot{X}_T^{s_p}}.$$

By Strichartz' estimates, we obtain

$$\|v_{i,\lambda_i}\|_{L_{t,x}^6} \lesssim \lambda_i^{-\frac{1}{6}-s_p} \|v_i\|_{\dot{X}_T^{s_p}}$$

for $i = 3, 4$, as well as

$$\|u_\mu\|_{L_{t,x}^6} \lesssim \mu^{-\frac{1}{6}} \|u_\mu\|_{V_{\text{KdV}}^2}.$$

Furthermore, since $q_2 > 2$ the bilinear estimate gives

$$\|v_{2,\lambda_2} v_{5,\lambda_5}\|_{L_{t,x}^{q_2}} \lesssim \lambda_2^{\frac{1}{2}-\frac{1}{q_2}-s_p} \lambda_5^{\frac{1}{2}-\frac{3}{q_2}-s_p} \|v_2\|_{\dot{X}_T^{s_p}} \|v_5\|_{\dot{X}_T^{s_p}}.$$

The product of the λ_i can be estimated by $\lambda_2^{\frac{1}{6}-\frac{3}{2(p+5)}} \lambda_5^{-1+\frac{3}{2(p+5)}-s_p}$. Note that the exponent of λ_2 is bigger than zero for all $p \geq 5$. \square

If we assume that the frequency μ is much greater than all other frequencies, then we can even prove the following Lemma.

Lemma 4.2. *Let $5 < p \in \mathbb{R}$ and $\lambda_2 \leq \dots \leq \lambda_5 \in 1.01^{\mathbb{Z}}$, $\mu \in 1.01^{\mathbb{Z}}$ and $1.1\lambda_5 \leq \mu$. There exists $r > 0$ independent of T such that for given $v_i, u \in \dot{X}_T^{s_p}$, $i = 0, \dots, 5$, we have*

$$\begin{aligned} & \left| \int |v_{0,\leq\lambda_2}|^{p-5} v_{1,\leq\lambda_2} v_{2,\leq\lambda_2} v_{3,\lambda_3} v_{4,\lambda_4} v_{5,\lambda_5} u_\mu dx dt \right| \\ & \leq r \lambda_2^{\frac{1}{15}} \lambda_5^{-\frac{1}{6}-s_p} \mu^{-\frac{9}{10}} \|v_0\|_{\dot{X}_T^{s_p}}^{p-5} \prod_{i=1}^5 \|v_i\|_{\dot{X}_T^{s_p}} \|u_\mu\|_{V_{\text{KdV}}^2}. \end{aligned}$$

Moreover, we may even replace one factor $\|v_i\|_{\dot{X}_T^{s_p}}$, $i = 3, 4$, on the right hand side by

$$\sup_{\lambda \in 1.01^{\mathbb{Z}}} \lambda^{\frac{1}{6}+s_p} \|v_{i,\lambda}\|_{L^6([0,T],\mathbb{R})}.$$

Proof. The proof is quite similar to the proof of Lemma 4.1. We consider

$$\left| \int |v_{0,\leq\lambda_2}|^{p-5} v_{1,\leq\lambda_2} v_{2,\leq\lambda_2} v_{3,\lambda_3} v_{4,\lambda_4} v_{5,\lambda_5} u_\mu dx dt \right|$$

and define the Hölder exponent $q = 5(p-3)$. Using Hölder's inequality we estimate

$$\|v_{0,\leq\lambda_2}\|_{L_{t,x}^q}^{p-5} \|v_{1,\leq\lambda_2}\|_{L_{t,x}^q} \|v_{2,\leq\lambda_2}\|_{L_{t,x}^q} \|v_{4,\lambda_4}\|_{L_{t,x}^6} \|v_{5,\lambda_5}\|_{L_{t,x}^6} \|v_{3,\lambda_3} u_\mu\|_{L_{t,x}^{15/7}}.$$

By Corollary 3.3 and the definition of $\dot{X}_T^{s_p}$, we obtain

$$\|v_{i,\leq\lambda_2}\|_{L_{t,x}^q} \lesssim \lambda_2^{\frac{1}{2}-\frac{4}{q}-s_p} \|v_i\|_{\dot{X}_T^{s_p}}, \quad i = 0, 1, 2.$$

Applying Strichartz' estimates yields

$$\|v_{i,\lambda_i}\|_{L_{t,x}^6} \lesssim \lambda_i^{-\frac{1}{6}-s_p} \|v_i\|_{\dot{X}_T^{s_p}}, \quad i = 4, 5.$$

Finally, the bilinear estimate provides

$$\|v_{3,\lambda_3} u_\mu\|_{L_{t,x}^{15/7}} \lesssim \lambda_3^{\frac{1}{30}-s_p} \mu^{-\frac{9}{10}} \|v_5\|_{\dot{X}_T^{s_p}} \|u_\mu\|_{V_{\text{KdV}}^2}.$$

The frequencies can be estimated by $\lambda_2^{\frac{1}{15}} \lambda_5^{-\frac{1}{6}-s_p} \mu^{-\frac{9}{10}}$.

Note that we may change the role of v_{3,λ_3} and v_{4,λ_4} in the calculation above and hence can also estimate v_{3,λ_3} in $L_{t,x}^6$. \square

5. PROOF OF THE THEOREM

In this section we are going to prove Theorem 1.2. The solution $\psi = v + w$ of (1) is constructed by studying the following equation

$$\begin{cases} \partial_t w + \partial_{xxx} w + \partial_x(|v + w|^{p-1}(v + w)) &= 0, \\ w(0, x) &= 0, \end{cases}$$

where v is a solution to the Airy equation (2).

Lemma 5.1. *Let $W \in \dot{X}_T^{s_p}$ and furthermore let r and κ_1 be the constants from Lemma 4.1 and Lemma 2.12, respectively. Under the assumptions of Theorem 1.2, we consider*

$$\begin{cases} \partial_t w + \partial_{xxx} w + \partial_x(|v + W|^{p-1}(v + W)) &= 0, \\ w(0, x) &= 0. \end{cases} \quad (7)$$

If w solves (7) and $\|W\|_{\dot{X}_T^{s_p}} \leq \alpha$, then there exists some $c > 0$ such that for

$$\alpha \leq \min \left\{ \kappa_1 r_0, \frac{1}{2cr\kappa_1^p r_0^{p-1}} \right\} \quad \text{and} \quad \delta_0 \leq \alpha,$$

it holds

$$\|w\|_{\dot{X}_T^{s_p}} \leq \alpha.$$

Proof. For $\tau \in \mathbb{R}$ and $\lambda \in 1.01^{\mathbb{Z}}$ we set $F_\lambda^\tau(u) = u_{<\lambda} + \tau u_\lambda$. Using that we define for $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (1.01^{\mathbb{Z}})^n$

$$\mathbf{F}_\lambda^\tau(u) = F_{\lambda_1}^{\tau_1} \circ \dots \circ F_{\lambda_n}^{\tau_n}(u).$$

One easily proves, that for $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n) \in [0, 1]^n$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (1.01^{\mathbb{Z}})^n$ and $\mu \in 1.01^{\mathbb{Z}}$ we have

$$|(\mathbf{F}_\lambda^\tau(u))_\mu| \leq 2^n |u_\mu| \quad \text{and} \quad \|\mathbf{F}_\lambda^\tau(u)\|_{\dot{X}_T^s} \leq 2^n \|u\|_{\dot{X}_T^s}, \quad s \in \mathbb{R}.$$

Furthermore, one trivially verifies

$$\mathbf{F}_\lambda^\tau(u + v) = \mathbf{F}_\lambda^\tau(u) + \mathbf{F}_\lambda^\tau(v).$$

Set $f_p(x) = |x|^{p-1}x$, then by the telescoping series we have

$$f_p(u) = \sum_{\lambda \in 1.01^{\mathbb{Z}}} (f_p(u_{\leq \lambda}) - f_p(u_{< \lambda})).$$

By a standard trick, using the fundamental theorem of calculus we get

$$\begin{aligned} f_p(u) &= \sum_{\lambda \in 1.01\mathbb{Z}} \int_0^1 f'_p(u_{<\lambda} + \tau(u_{\leq\lambda} - u_{<\lambda})) d\tau(u_{\leq\lambda} - u_{<\lambda}) \\ &= \sum_{\lambda \in 1.01\mathbb{Z}} \int_0^1 f'_p(F_\lambda^\tau(u)) d\tau u_\lambda \end{aligned}$$

In the sequel we use a more compact notation and write

$$\mathbf{u}_\lambda^\tau := \mathbf{F}_\lambda^\tau(u).$$

Reapplying this method three times, we get for $\boldsymbol{\lambda}_i = (\lambda_i, \dots, \lambda_5)$ and $\boldsymbol{\tau}_i = (\tau_i, \dots, \tau_5)$, $i = 2, \dots, 5$,

$$f_p(u) = \sum_{\substack{\lambda_2 \leq \dots \leq \lambda_5 \\ \lambda_i \in 1.01\mathbb{Z}}} \int_{[0,1]^4} f_p^{(4)}(\mathbf{u}_{\lambda_2}^{\tau_2}) \mathbf{u}_{\lambda_3, \lambda_2}^{\tau_3} \mathbf{u}_{\lambda_4, \lambda_3}^{\tau_4} \mathbf{u}_{\lambda_5, \lambda_4}^{\tau_5} u_{\lambda_5} d\boldsymbol{\tau}_2.$$

That is in our context

$$|u|^{p-1}u = c \sum_{\substack{\lambda_2 \leq \dots \leq \lambda_5 \\ \lambda_i \in 1.01\mathbb{Z}}} \int_{[0,1]^4} |\mathbf{u}_{\lambda_2}^{\tau_2}|^{p-5} \mathbf{u}_{\lambda_2}^{\tau_2} \mathbf{u}_{\lambda_3, \lambda_2}^{\tau_3} \mathbf{u}_{\lambda_4, \lambda_3}^{\tau_4} \mathbf{u}_{\lambda_5, \lambda_4}^{\tau_5} u_{\lambda_5} d\boldsymbol{\tau}_2,$$

where $c = p(p-1)(p-2)(p-3)(p-4)$.

Let w be a solution to (7). By Lemma 2.12 it suffices to show

$$\sup_{\mu \in 1.01\mathbb{Z}} \mu^{s_p} \left\| \int_{-\infty}^t e^{-(t-s)\partial_x^3} (\partial_x(|v+W|^{p-1}(v+W)))_\mu(s) ds \right\|_{U_{\text{KdV}}^2} \leq \frac{\alpha}{\kappa_1}.$$

By duality (cf. Corollary 2.7), it suffices to show that for each $\mu \in 1.01\mathbb{Z}$ we have

$$\frac{\mu^{s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \left| \int \partial_x(|v+W|^{p-1}(v+W)) u_\mu dx dt \right| \leq \frac{\alpha}{\kappa_1}.$$

If we apply the calculation above once, we can rewrite the modulus of the integral as

$$\begin{aligned} \mathbf{S}_1 + \mathbf{S}_2 &:= \sum_{\lambda_5: 1.1\lambda_5 > \mu} \left| \int \partial_x(|\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5}|^{p-1}(v+W)_{\lambda_5}) u_\mu dx dt d\boldsymbol{\tau}_5 \right| \\ &+ \sum_{\lambda_5: 1.1\lambda_5 \leq \mu} \left| \int \partial_x(|\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5}|^{p-1}(v+W)_{\lambda_5}) u_\mu dx dt d\boldsymbol{\tau}_5 \right|. \end{aligned}$$

First, we consider the sum \mathbf{S}_1 . We integrate by parts, apply the calculation above to $|\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5}|^{p-1}$ and hence have to bound

$$\begin{aligned} & \frac{\mu^{1+s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \sum_{\substack{\lambda_2 \leq \dots \leq \lambda_5 \\ \lambda_5: 1.1\lambda_5 > \mu}} \left| \int |\mathbf{v}_{\lambda_2}^{\tau_2} + \mathbf{W}_{\lambda_2}^{\tau_2}|^{p-5} (\mathbf{v}_{\lambda_2}^{\tau_2} + \mathbf{W}_{\lambda_2}^{\tau_2}) \right. \\ & \quad \times (\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3})_{\lambda_2} \cdots (\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4} (v + W)_{\lambda_5} \frac{\partial_x}{\mu} u_\mu dx dt d\tau_2 \Big|. \end{aligned}$$

Note that since the spaces V_{KdV}^2 are based on L^2 , the operator $\frac{\partial_x}{\mu}$ is bounded. We expand the factor $(\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4}$. For $\mathbf{v}_{\lambda_5, \lambda_4}^{\tau_5}$ we apply Lemma 4.1 and keep this factor in $L_{t,x}^6$. For $\mathbf{W}_{\lambda_5, \lambda_4}^{\tau_5}$ we apply Lemma 4.1 and estimate all terms in $\dot{X}_T^{s_p}$. Hence, after summing over the frequencies, we obtain that \mathbf{S}_1 is less than

$$\begin{aligned} & \int_{[0,1]^4} r \|\mathbf{v}_{\lambda_2}^{\tau_2} + \mathbf{W}_{\lambda_2}^{\tau_2}\|_{\dot{X}_T^{s_p}}^{p-4} \|\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}\|_{\dot{X}_T^{s_p}} \|\mathbf{v}_{\lambda_4}^{\tau_4} + \mathbf{W}_{\lambda_4}^{\tau_4}\|_{\dot{X}_T^{s_p}} \|v + W\|_{\dot{X}_T^{s_p}} \\ & \quad \times \left(\sup_{\lambda \in 1.01\mathbb{Z}} \lambda^{\frac{1}{6}+s_p} \|\mathbf{v}_{\lambda_5, \lambda}^{\tau_5}\|_{L^6([0,T], \mathbb{R})} + \|\mathbf{W}_{\lambda_5}^{\tau_5}\|_{\dot{X}_T^{s_p}} \right) d\tau_2. \end{aligned}$$

By the properties of $\mathbf{F}_\lambda^\tau(\cdot)$, we may estimate this by

$$cr \|v + W\|_{\dot{X}_T^{s_p}}^{p-1} \left(\sup_{\lambda \in 1.01\mathbb{Z}} \lambda^{\frac{1}{6}+s_p} \|v_\lambda\|_{L^6([0,T], \mathbb{R})} + \|W\|_{\dot{X}_T^{s_p}} \right).$$

Using the bounds given in Theorem 1.2 and $\alpha \leq \kappa_1 r_0$ we may estimate this by

$$cr (r_0 \kappa_1)^{p-1} (\delta_0 + \alpha)$$

for some $c > 0$. Since $\delta_0 \leq \alpha$ and $\alpha \leq \frac{1}{2cr\kappa_1^p r_0^{p-1}}$ we obtain

$$cr (r_0 \kappa_1)^{p-1} (\delta_0 + \alpha) \leq \frac{\alpha}{\kappa_1},$$

which implies the desired estimate $\|w\|_{\dot{X}_T^{s_p}} \leq \alpha$ for \mathbf{S}_1 .

Now, we consider \mathbf{S}_2 . Note that $\mathbf{S}_2 = 0$ if $p = 5$, since the frequencies do not sum up to zero. Hence we may assume $p > 5$ in the following. In order to estimate \mathbf{S}_2 , we decompose

$$\begin{aligned} & |\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5}|^{p-1} (v + W)_{\lambda_5} \\ &= \sum_{\lambda_3 \leq \lambda_4 \leq \lambda_5} \int |\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}|^{p-3} (\mathbf{v}_{\lambda_4}^{\tau_4} + \mathbf{W}_{\lambda_4}^{\tau_4})_{\lambda_3} (\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4} (v + W)_{\lambda_5} d\tau_3. \end{aligned}$$

Differentiating this term with respect to x yields

$$\begin{aligned}
& c\lambda_3 |\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}|^{p-5} (\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}) \frac{\partial_x}{\lambda_3} (\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}) (\mathbf{v}_{\lambda_4}^{\tau_4} + \mathbf{W}_{\lambda_4}^{\tau_4})_{\lambda_3} (\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4} (v + W)_{\lambda_5} \\
& + \lambda_3 |\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}|^{p-3} \frac{\partial_x}{\lambda_3} (\mathbf{v}_{\lambda_4}^{\tau_4} + \mathbf{W}_{\lambda_4}^{\tau_4})_{\lambda_3} (\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4} (v + W)_{\lambda_5} \\
& + \lambda_4 |\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}|^{p-3} (\mathbf{v}_{\lambda_4}^{\tau_4} + \mathbf{W}_{\lambda_4}^{\tau_4})_{\lambda_3} \frac{\partial_x}{\lambda_3} (\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4} (v + W)_{\lambda_5} \\
& + \lambda_5 |\mathbf{v}_{\lambda_3}^{\tau_3} + \mathbf{W}_{\lambda_3}^{\tau_3}|^{p-3} (\mathbf{v}_{\lambda_4}^{\tau_4} + \mathbf{W}_{\lambda_4}^{\tau_4})_{\lambda_3} (\mathbf{v}_{\lambda_5}^{\tau_5} + \mathbf{W}_{\lambda_5}^{\tau_5})_{\lambda_4} \frac{\partial_x}{\lambda_5} (v + W)_{\lambda_5}.
\end{aligned}$$

We estimate all these terms exactly as for \mathbf{S}_1 , but using Lemma 4.2 instead of Lemma 4.1. Note that the additional factor λ_i on each term ensures that the summation over the frequencies converges. Note also that the operator $\frac{\partial_x}{\lambda_i}$ does not play a role, since the V_{KdV}^2 spaces are based on L^2 . By the same argument as before, we can bound \mathbf{S}_2 by $\frac{\alpha}{\kappa_1}$ as before. \square

Proof of Theorem 1.2. In order to prove Theorem 1.2, we use a fixed point argument to show existence and uniqueness. Let

$$\alpha \leq \min \left\{ \kappa_1 r_0, \frac{1}{2cr\kappa_1^p r_0^{p-1}} \right\}, \quad \delta_0 \leq \alpha \quad \text{and} \quad \|w_0\|_{\dot{X}_T^{sp}} < \alpha.$$

Furthermore, let

$$\begin{cases} \partial_t w_1 + \partial_{xxx} w_1 + \partial_x (|v + w_0|^{p-1} (v + w_0)) & = 0, \\ w_1(0, x) & = 0, \end{cases}$$

and

$$\begin{cases} \partial_t w_2 + \partial_{xxx} w_2 + \partial_x (|v + w_1|^{p-1} (v + w_1)) & = 0, \\ w_2(0, x) & = 0, \end{cases}$$

be two iteration steps. Note that Lemma 5.1 ensures that $\|w_1\|_{\dot{X}_T^{sp}} < \alpha$ as well. We have to show that there exists $q \in (0, 1)$ such that

$$\|w_2 - w_1\|_{\dot{X}_T^{sp}} \leq q \|w_1 - w_0\|_{\dot{X}_T^{sp}}.$$

By Lemma 2.12 and duality, it suffices to replace the left hand side by

$$\kappa_1 \frac{\mu^{sp}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \left| \int \partial_x (|v + w_0|^{p-1} (v + w_0) - |v + w_1|^{p-1} (v + w_1)) u_\mu dx dt \right|.$$

For brevity we define $\omega_0 = v + w_0$ and $\omega_1 = v + w_1$. Similar as in the proof of Lemma 5.1 we may write

$$|\omega_0|^{p-1} \omega_0 - |\omega_1|^{p-1} \omega_1 = \sum_{\lambda_5 \in 1.01\mathbb{Z}} \int |\omega_0^{\tau_5}_{\lambda_5}|^{p-1} \omega_{0, \lambda_5} - |\omega_1^{\tau_5}_{\lambda_5}|^{p-1} \omega_{1, \lambda_5} d\tau_5.$$

Again, we split the sum into two parts, such that

$$\begin{aligned} \mathbf{S}_1 + \mathbf{S}_2 = & \sum_{\lambda_5: 1.1\lambda_5 > \mu} \left| \int \partial_x (|\omega_{\mathbf{0}\lambda_5}^{\tau_5}|^{p-1} \omega_{0,\lambda_5} - |\omega_{\mathbf{1}\lambda_5}^{\tau_5}|^{p-1} \omega_{1,\lambda_5}) u_\mu dx dt d\tau_5 \right| \\ & + \sum_{\lambda_5: 1.1\lambda_5 \leq \mu} \left| \int \partial_x (|\omega_{\mathbf{0}\lambda_5}^{\tau_5}|^{p-1} \omega_{0,\lambda_5} - |\omega_{\mathbf{1}\lambda_5}^{\tau_5}|^{p-1} \omega_{1,\lambda_5}) u_\mu dx dt d\tau_5 \right|. \end{aligned}$$

First, we consider \mathbf{S}_1 . Similar to the previous Lemma, we integrate by parts such that the derivative turns into a factor μ , and we decompose

$$\int_{[0,1]} |\omega_{\mathbf{0}\lambda_5}^{\tau_5}|^{p-1} \omega_{0,\lambda_5} - |\omega_{\mathbf{1}\lambda_5}^{\tau_5}|^{p-1} \omega_{1,\lambda_5} d\tau_5$$

to

$$\begin{aligned} \sum_{\lambda_2 \leq \dots \leq \lambda_5} \int_{[0,1]^4} & |\omega_{\mathbf{0}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{0}\lambda_2}^{\tau_2} \omega_{\mathbf{0}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{0}\lambda_4,\lambda_3}^{\tau_4} \omega_{\mathbf{0}\lambda_5,\lambda_4}^{\tau_5} \omega_{0,\lambda_5} \\ & - |\omega_{\mathbf{1}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{1}\lambda_2}^{\tau_2} \omega_{\mathbf{1}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{1}\lambda_4,\lambda_3}^{\tau_4} \omega_{\mathbf{1}\lambda_5,\lambda_4}^{\tau_5} \omega_{1,\lambda_5} d\tau_2. \end{aligned}$$

The integrand may be written as

$$\begin{aligned} & |\omega_{\mathbf{0}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{0}\lambda_2}^{\tau_2} \omega_{\mathbf{0}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{0}\lambda_4,\lambda_3}^{\tau_4} \omega_{\mathbf{0}\lambda_5,\lambda_4}^{\tau_5} (\omega_{0,\lambda_5} - \omega_{1,\lambda_5}) \\ & + |\omega_{\mathbf{0}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{0}\lambda_2}^{\tau_2} \omega_{\mathbf{0}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{0}\lambda_4,\lambda_3}^{\tau_4} (\omega_{\mathbf{0}\lambda_5,\lambda_4}^{\tau_5} - \omega_{\mathbf{1}\lambda_5,\lambda_4}^{\tau_5}) \omega_{1,\lambda_5} \\ & + \dots \\ & + (|\omega_{\mathbf{0}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{0}\lambda_2}^{\tau_2} - |\omega_{\mathbf{1}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{1}\lambda_2}^{\tau_2}) \omega_{\mathbf{1}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{1}\lambda_4,\lambda_3}^{\tau_4} \omega_{\mathbf{1}\lambda_5,\lambda_4}^{\tau_5} \omega_{1,\lambda_5} \\ & =: S_1 + \dots + S_5 \end{aligned}$$

Using the fundamental theorem of calculus, we can further manipulate the last term S_5 to get

$$\begin{aligned} S_5 = c \int_0^1 & |\omega_{\mathbf{0}\lambda_2}^{\tau_2} + \tau(\omega_{\mathbf{1}\lambda_2}^{\tau_2} - \omega_{\mathbf{0}\lambda_2}^{\tau_2})|^{p-5} d\tau (\omega_{\mathbf{0}\lambda_2}^{\tau_2} - \omega_{\mathbf{1}\lambda_2}^{\tau_2}) \\ & \times \omega_{\mathbf{1}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{1}\lambda_4,\lambda_3}^{\tau_4} \omega_{\mathbf{1}\lambda_5,\lambda_4}^{\tau_5} \omega_{1,\lambda_5}. \end{aligned}$$

We split S_1 into two terms by expanding $\omega_{\mathbf{0}\lambda_5,\lambda_4}^{\tau_5}$:

$$\begin{aligned} S_1 = & |\omega_{\mathbf{0}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{0}\lambda_2}^{\tau_2} \omega_{\mathbf{0}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{0}\lambda_4,\lambda_3}^{\tau_4} \mathbf{v}_{\lambda_5,\lambda_4}^{\tau_5} (w_{0,\lambda_5} - w_{1,\lambda_5}) \\ & + |\omega_{\mathbf{0}\lambda_2}^{\tau_2}|^{p-5} \omega_{\mathbf{0}\lambda_2}^{\tau_2} \omega_{\mathbf{0}\lambda_3,\lambda_2}^{\tau_3} \omega_{\mathbf{0}\lambda_4,\lambda_3}^{\tau_4} \mathbf{w}_{\lambda_5,\lambda_4}^{\tau_5} (w_{0,\lambda_5} - w_{1,\lambda_5}). \end{aligned}$$

For the first term we estimate $\mathbf{v}_{\lambda_5, \lambda_4}^{\tau_5}$ in $L_{t,x}^6$, and for the second term we estimate all factors in $\dot{X}_T^{s_p}$. Hence, by Lemma 4.1

$$\frac{\mu^{1+s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \sum_{\substack{\lambda_2 \leq \dots \leq \lambda_5 \\ \lambda_5: 1.1\lambda_5 \leq \mu}} \left| \int S_1 \frac{\partial_x}{\mu} u_\mu dx dt d\tau_2 \right| \lesssim r(r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) \|w_0 - w_1\|_{\dot{X}_T^{s_p}}.$$

Analogously, we expand either $\omega_{\mathbf{0}_{\lambda_4, \lambda_3}}^{\tau_4}$ or $\omega_{\mathbf{1}_{\lambda_5, \lambda_4}}^{\tau_5}$ in S_2, \dots, S_5 . For each S_i , $i = 2, \dots, 5$, either the expanded term depends on v , then we choose to estimate this factor in $L_{t,x}^6$, or we estimate all factors in $\dot{X}_T^{s_p}$. Thus, by Lemma 4.1 we estimate for $i = 2, \dots, 5$

$$\frac{\mu^{1+s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \sum_{\substack{\lambda_2 \leq \dots \leq \lambda_5 \\ \lambda_5: 1.1\lambda_5 \leq \mu}} \left| \int S_i \frac{\partial_x}{\mu} u_\mu dx dt d\tau_2 \right| \lesssim r(r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) \|w_0 - w_1\|_{\dot{X}_T^{s_p}}.$$

All in all, we obtain

$$\kappa_1 \frac{\mu^{s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \mathbf{S}_1 \leq cr \kappa_1 (r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) \|w_0 - w_1\|_{\dot{X}_T^{s_p}}.$$

Now, we may choose α (and hence δ_0) small enough such that

$$cr \kappa_1 (r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) < \frac{1}{2},$$

which gives the desired estimate for \mathbf{S}_1 .

Now, we consider

$$\mathbf{S}_2 = \sum_{\lambda_5: 1.1\lambda_5 \leq \mu} \left| \int \partial_x (|\omega_{\mathbf{0}_{\lambda_5}}^{\tau_5}|^{p-1} \omega_{0, \lambda_5} - |\omega_{\mathbf{1}_{\lambda_5}}^{\tau_5}|^{p-1} \omega_{1, \lambda_5}) u_\mu dx dt d\tau_5 \right|.$$

Note that if $p = 5$ then $\mathbf{S}_2 = 0$ by the same argument as in Lemma 5.1.

Hence, we may assume $p > 5$. We decompose

$$\begin{aligned} & \int_{[0,1]} |\omega_{\mathbf{0}_{\lambda_5}}^{\tau_5}|^{p-1} \omega_{0, \lambda_5} - |\omega_{\mathbf{1}_{\lambda_5}}^{\tau_5}|^{p-1} \omega_{1, \lambda_5} d\tau_5 \\ &= \sum_{\lambda_3 \leq \lambda_4 \leq \lambda_5} \int |\omega_{\mathbf{0}_{\lambda_3}}^{\tau_3}|^{p-3} \omega_{\mathbf{0}_{\lambda_4, \lambda_3}}^{\tau_4} \omega_{\mathbf{0}_{\lambda_5, \lambda_4}}^{\tau_5} \omega_{0, \lambda_5} - |\omega_{\mathbf{1}_{\lambda_3}}^{\tau_3}|^{p-3} \omega_{\mathbf{1}_{\lambda_4, \lambda_3}}^{\tau_4} \omega_{\mathbf{1}_{\lambda_5, \lambda_4}}^{\tau_5} \omega_{1, \lambda_5} d\tau_3 \\ &= \sum_{\lambda_3 \leq \lambda_4 \leq \lambda_5} \int |\omega_{\mathbf{0}_{\lambda_3}}^{\tau_3}|^{p-3} \omega_{\mathbf{0}_{\lambda_4, \lambda_3}}^{\tau_4} \omega_{\mathbf{0}_{\lambda_5, \lambda_4}}^{\tau_5} (\omega_{0, \lambda_5} - \omega_{1, \lambda_5}) \\ & \quad + \dots \\ & \quad + (|\omega_{\mathbf{0}_{\lambda_3}}^{\tau_3}|^{p-3} - |\omega_{\mathbf{1}_{\lambda_3}}^{\tau_3}|^{p-3}) \omega_{\mathbf{1}_{\lambda_4, \lambda_3}}^{\tau_4} \omega_{\mathbf{1}_{\lambda_5, \lambda_4}}^{\tau_5} \omega_{1, \lambda_5} d\tau_3 \\ &=: \sum_{\lambda_3 \leq \lambda_4 \leq \lambda_5} \int S_1 + \dots + S_4 d\tau_3 \end{aligned}$$

If we differentiate S_1 with respect to x , then we are able to apply Lemma 4.2 and since we obtain an additional factor λ_i , we get

$$\begin{aligned}
& \kappa_1 \frac{\mu^{s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \sum_{\substack{\lambda_3 \leq \lambda_4 \leq \lambda_5 \\ \lambda_5: 1.1\lambda_5 \leq \mu}} \left| \int S_1 u_\mu dx dt d\tau_3 \right| \\
& \leq cr \|\omega_0\|_{\dot{X}_T^{s_p}}^{\tau_3} \|\omega_0\|_{\dot{X}_T^{s_p}}^{\tau_4} \left(\sup_{\lambda \in 1.01\mathbb{Z}} \lambda^{\frac{1}{6}+s_p} \|\mathbf{v}_{\lambda_5, \lambda}^{\tau_5}\|_{L([0, T], \mathbb{R})} + \|\mathbf{w}_0\|_{\dot{X}_T^{s_p}}^{\tau_5} \right) \\
& \quad \times \|\omega_0 - \omega_1\|_{\dot{X}_T^{s_p}} \\
& \leq cr \kappa_1 (r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) \|w_0 - w_1\|_{\dot{X}_T^{s_p}}.
\end{aligned}$$

We can treat S_2 and S_3 analogously. Now, we consider S_4 . Applying the fundamental theorem of calculus, we obtain for $\Omega(\tau) = \omega_0^{\tau_3} + \tau(\omega_1^{\tau_3} - \omega_0^{\tau_3})$:

$$S_4 = c \int_0^1 |\Omega(\tau)|^{p-5} \Omega(\tau) d\tau (\omega_0^{\tau_3} - \omega_1^{\tau_3}) \omega_1^{\tau_4} \omega_1^{\tau_5} \omega_{1, \lambda_5}.$$

Differentiating this term with respect to x yields a sum of 5 terms, each of which can be estimated using Lemma 4.2 as above. All in all we obtain

$$\kappa_1 \frac{\mu^{s_p}}{\|u_\mu\|_{V_{\text{KdV}}^2}} \mathbf{S}_2 \leq cr \kappa_1 (r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) \|w_0 - w_1\|_{\dot{X}_T^{s_p}}.$$

By possibly choosing α smaller again, we have

$$cr \kappa_1 (r_0 \kappa_1)^{p-2} (\delta_0 + \alpha) < \frac{1}{2}.$$

Thus, for small enough α , we have a contraction and Banach fixed-point theorem gives existence and uniqueness. \square

The following proof of Corollary 1.3 is an observation of Koch and Marzuola in [8, p. 175-176].

Proof of Corollary 1.3. By Strichartz' estimates for linear KdV and Lemma 2.12 we have for v given as in (2) and $0 < T \leq \infty$ that

$$\sup_{\lambda \in 1.01\mathbb{Z}} \lambda^{\frac{1}{6}+s_p} \|v_\lambda\|_{L^6([0, T], L^6(\mathbb{R}))} \leq \kappa_0 \|v\|_{\dot{X}_T^{s_p}} \leq \kappa_0 \kappa_1 \|\psi_0\|_{\dot{B}_\infty^{s_p, 2}} \leq \delta_0(1).$$

Since this estimate holds true for all $0 < T \leq \infty$, we may apply Theorem 1.2 with $T = \infty$ to obtain global existence. \square

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